

Improved bound on facial parity edge coloring

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Abstract

A *facial parity edge coloring* of a 2-edge connected plane graph is an edge coloring where no two consecutive edges of a facial walk of any face receive the same color. Additionally, for every face f and every color c either no edge or an odd number of edges incident to f are colored by c . Czap, Jendrol', Kardoš and Soták [3] showed that every 2-edge connected plane graph admits a facial parity edge coloring with at most 20 colors. We improve this bound to 16 colors.

Keywords: Facial parity edge coloring, planar graph, odd subgraph.

1 Introduction

A *facial parity edge coloring* or an *FPE-coloring* of a 2-edge connected plane graph is an edge coloring where no two consecutive edges of a facial walk of any face receive the same color. Additionally, for every face f and every color c either no edge or an odd number of edges incident to f are colored by c . The minimum number of colors such that an FPE-coloring of a graph G exists is called the *facial parity chromatic index*, $\chi_{fp}(G)$.

In [2], Czap, Jendrol', and Kardoš defined the FPE-coloring of plane graphs and proved that 92 colors suffice to color any 2-edge connected plane graph. Recently, Czap et al. [3] improved the upper bound.

Theorem 1 (Czap, Jendrol', Kardoš, and Soták). *Let G be a 2-edge connected plane graph. Then,*

$$\chi_{fp}(G) \leq 20.$$

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They also showed that if G is 3-edge connected, 12 colors suffice, and in case when G is a 4-edge connected plane graph, it has an FPE-coloring with at most 9 colors. In addition, Czap [1] showed that facial parity chromatic index of outerplanar graphs is at most 15 and presented the example, two pentagons with one vertex identified, which requires 10 colors for an FPE-coloring (see Fig. 1). In this paper we improve the upper

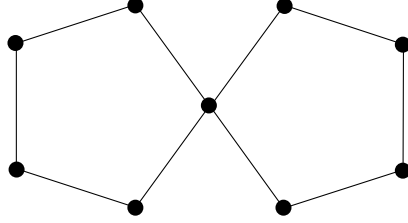


Figure 1: The graph that needs at least 10 colors for an FPE-coloring.

bound for facial parity chromatic index of any 2-edge connected plane graph.

Theorem 2. *Let G be a 2-edge connected plane graph. Then,*

$$\chi_{fp}(G) \leq 16.$$

2 Proof of Theorem 2

Before we prove Theorem 2, we present some definitions and the notation. A *dual* $G^* = (V^*, E^*, F^*)$ of a plane graph $G = (V, E, F)$ is the graph with $V^* = V$ and two vertices f, f' of G^* are connected by an edge e^* if the faces f and f' are incident to an edge e in G . We say that the edges e^* and e are *associated*. An H -block of a graph G is a block in G isomorphic to the graph H . Let V be some subset of vertices of a graph G . As usual, $G[V]$ is a subgraph of G induced by the vertices of V .

The *facial walk* of a face f is the shortest closed walk containing all the edges from the boundary of f . Two edges are *face-adjacent* if they are consecutive on some facial walk. An edge coloring is *facially proper* if the face-adjacent edges receive distinct colors. It is easy to see that a facially proper edge coloring is equivalent to a vertex coloring of the medial graph of G , which is planar and therefore 4-colorable by the Four Color Theorem. We say that an edge coloring of G is *facially-odd* if it is facially proper and for any pair of adjacent faces f and f' holds that the number of common edges colored by the same color c is odd or zero, for every color c .

In [3], the authors proved that for any connected plane graph there is a facially-odd edge coloring with at most 5 colors. The bound is best possible if the graph contains blocks isomorphic to C_5 , a cycle on 5 vertices. We say that an edge coloring is *quasi-facially-odd* if it is facially-odd on all the edges, except for the edges of C_5 -blocks. The edges of every C_5 -block are colored by 4 colors such that the color that appears twice is assigned to the edges that are not face-adjacent.

Lemma 3. *Every connected plane graph admits a quasi-facially-odd edge coloring with at most 4 colors.*

Proof. Suppose, for a contradiction, that G is a minimal counterexample to the lemma, i.e., a connected plane graph with the minimum number of edges that does not admit a quasi-facially-odd edge coloring with at most 4 colors.

First, we show that G is 2-connected. Suppose, to the contrary, that v is a cut-vertex in G . Let C_1 be one of the components of $G - v$ and C_2 the subgraph comprised of the remaining components. By the minimality of G , there exist quasi-facially-odd edge colorings φ_1, φ_2 with at most 4 colors of the graphs $C'_1 = G[V(C_1) \cup \{v\}]$ and $C'_2 = G[V(C_2) \cup \{v\}]$. Let (e_1, e'_1) and (e_2, e'_2) be the two pairs of face-adjacent edges in G , where $e_1, e_2 \in C'_1$ and $e'_1, e'_2 \in C'_2$, see Figure 2. Notice that the edge e_1 may

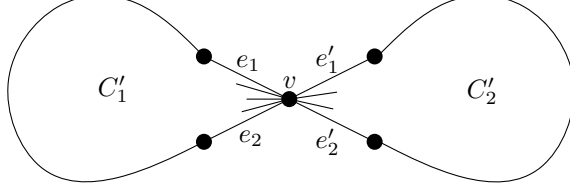


Figure 2: The face-adjacent edges of C'_1 and C'_2 .

be equal to e_2 , and e'_1 may be equal to e'_2 . By permuting the colors of φ_2 such that $\varphi_1(e_1) \neq \varphi_2(e'_1)$ and $\varphi_1(e_2) \neq \varphi_2(e'_2)$ we obtain a quasi-facially-odd edge coloring of G . Obviously, such a permutation exists. Thus, G is 2-connected. In particular, G does not contain a C_5 -block, unless G is isomorphic to C_5 .

If any pair of faces in G has at most one edge in common, every facially proper edge coloring is also quasi-facially-odd. Thus, we may assume that there is at least one pair of faces in G that has at least two edges in common. Let f and f' be two faces with k common edges, e_1, e_2, \dots, e_k , $k \geq 2$, appearing in the given order on the facial walk of f . First, we consider the case when two consecutive edges, e_i and e_{i+1} , form a nontrivial edge-cut in G , i.e., each of the two components, D_1 and D_2 , of $G - \{e_i, e_{i+1}\}$ contains at least two vertices. Let $H_1 = G/D_2$, i.e., H is constructed from G by contracting D_2 into a vertex, and similarly let $H_2 = G/D_1$.

In order to obtain a quasi-facially-odd edge coloring of G , we consider two cases. Suppose first that one of the subgraphs H_1 and H_2 , say H_1 , is isomorphic to C_5 (see Figure 3). Observe that in this case H_2 is not isomorphic to C_5 , by the choice of e_i and e_{i+1} . By the minimality of G , there exists a quasi-facially-odd edge coloring ρ of

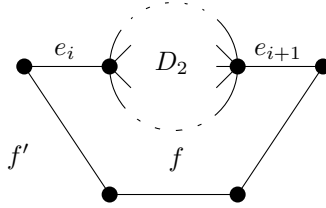


Figure 3: An example of the graph G where H_1 is a 5-cycle.

H_2/e_{i+1} . We show that ρ can be extended to G . Let e_i, c_1, c_2, c_3 , and e_{i+1} be the consecutive edges of H_1 . We assign the color $\rho(e_i)$ to e_{i+1} and c_3 , and color c_2 and c_4 with the two of the three remaining colors. Hence, extending ρ to G .

Suppose now that neither H_1 nor H_2 is isomorphic to C_5 . By the minimality of G , there are quasi-facially-odd edge colorings ρ_1 and ρ_2 with at most 4 colors of H_1 and H_2 , respectively. By permuting the colors of the edges of H_2 such that $\rho_1(e_i) = \rho_2(e_i)$ and $\rho_1(e_{i+1}) = \rho_2(e_{i+1})$, we obtain a quasi-facially-odd edge coloring of G . Such a

permutation always exists, since the edges e_i and e_{i+1} receive distinct colors in both graphs, H_1 and H_2 , due to the face-adjacency.

Hence, we may assume that no two consecutive common edges of f and f' form a nontrivial edge-cut. That means that the edges e_1, e_2, \dots, e_k form a path or possibly a cycle if f and f' are the only faces in G , since G is 2-connected. We consider the subcases regarding the size of k .

- $k = 2$. By the minimality, there is a quasi-facially-odd edge coloring of G/e_2 with 4 colors. Notice that at most three colors are forbidden for the only noncolored edge e_2 , the colors of the face-adjacent edges. Hence we can color it.
- $k = 3$. By the minimality, there is a quasi-facially-odd edge coloring of G/e_2 with 4 colors. Only the colors of e_1 and e_3 are forbidden for e_2 . Hence, we color e_2 with one of the two remaining colors.
- $k = 4$. Similarly as in the previous case, we color $G/\{e_2, e_3\}$ with 4 colors. The colors of e_1 and e_4 are forbidden for e_2 and e_3 . We color them with the two remaining colors.
- $k = 5$. Since G is 2-connected, the edges e_1, e_2, \dots, e_5 do not form a C_5 -block in G , unless $G = C_5$. In case when $G = C_5$, we color the edge e_1 and e_3 by color 1 and the remaining three edges by the remaining three colors. Otherwise, we color $G/\{e_2, \dots, e_5\}$ with 4 colors. Then, we use the color of e_1 to color e_3 and e_5 , since e_5 is not face-adjacent with e_1 , and two of the three remaining colors to color e_2 and e_4 .
- $k \geq 6$. If G is not a 9-cycle, we color $G/\{e_2, \dots, e_5\}$ with 4 colors. Then, we use the color of e_1 to color e_3 and e_5 and the color of e_6 to color e_2 and e_4 . Otherwise, $G = C_9$ and we color the nine edges by the colors 1, 2, and 3 alternatingly.

It is easy to see that every color appears odd times or does not appear at all on the edges e_1, e_2, \dots, e_k . Hence, G admits a quasi-facially-odd edge coloring with at most 4 colors, a contradiction. \square

As a corollary of Lemma 3, we infer that if the plane graph is 2-connected and not isomorphic to C_5 , it admits a facially-odd edge coloring with at most 4 colors.

An *odd graph* is a graph where all the vertices have odd or zero degree. An *odd edge coloring* of a graph G is a coloring (not necessarily proper) of the edges of G such that every color class induces an odd subgraph. The minimum number of colors for which an odd edge coloring of G exists is called the *odd chromatic index* of G , $\chi'_o(G)$. Obviously, every odd graph admits an odd edge coloring by one color. In 1991, Pyber [4] proved the following theorem.

Theorem 4 (Pyber). *Let G be a simple graph. Then,*

$$\chi'_o(G) \leq 4.$$

The tight bound is realized by W_4 , the wheel of 4 spokes (the left graph in Figure 4). For an odd edge coloring of a multigraph we may need more than 4 colors, e.g., the right graph in Figure 4 needs 6 colors. However, if the given multigraph has some special properties, 4 subgraphs suffice. The following simple result is mentioned in [4] but we prove it here for the sake of completeness.

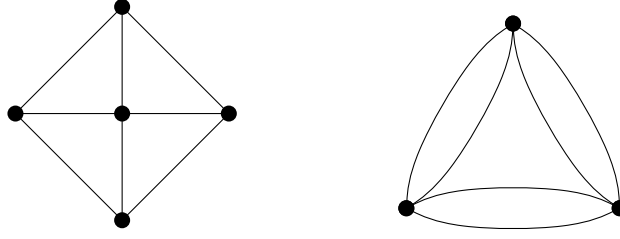


Figure 4: A simple graph with the odd chromatic index equal to 4, and a multigraph with the odd chromatic index equal to 6.

Lemma 5. *Let F be a forest. Then,*

$$\chi'_o(F) \leq 2.$$

Proof. It suffices to show that every tree T has the odd chromatic index at most 2. We will construct an odd edge coloring of T with at most two colors. Let v be an arbitrary vertex of T with degree $d \geq 1$ (otherwise T has no edges and so $\chi'_o(T) = 0$). If d is odd, we color all the edges incident to v by 1, otherwise we color $d - 1$ incident edges by 1 and the remaining with 2.

Now, we continue by coloring the edges incident to the neighbors of v . Each neighbor u already has one incident edge colored by some color, say 1, thus we color all the other incident edges of u with 1, unless $d(u)$ is even. In the latter case we color them by color 2. We repeat the same procedure on the vertices that have only one incident edge colored and eventually we obtain an odd edge coloring of T with at most two colors. \square

A k -bridge in a graph G is a collection of $k \geq 1$ edges between two vertices whose deletion results in an increased number of components in G .

Lemma 6. *Let G be a loopless multigraph with a k -bridge. Then,*

$$\chi'_o(G) \leq 4.$$

Proof. Suppose, for a contradiction, that G is a loopless multigraph with a k -bridge having the minimum number of edges with $\chi'_o(G) \geq 5$. It immediately follows that G is connected.

We construct an odd edge coloring of G with at most 4 colors in the following way. Let u and v be the end-vertices of the k -bridge in G and let G_u and G_v be the two components of G with the edges of the k -bridge removed. Note that one or both components may consist of only one vertex u or v , respectively. We will separately color the graphs $G'_u = G[V(G_u) \cup \{v\}]$ and $G'_v = G[V(G_v) \cup \{u\}]$ by at most 4 colors and then show that the obtained colorings induce an odd edge coloring of G with at most 4 colors.

In the sequel, we color G'_u and the analogous procedure may be used to color G'_v . Let T be a spanning tree of G'_u . Note that the vertex v is a leaf in T . Consider two cases regarding the number of vertices n of G'_u . If n is even, then there is also an even number of vertices of even degree $v_1, v_2, \dots, v_{2\ell}$ in G'_u . Let P_i be a path between v_{2i-1} and v_{2i} in T for $i \in \{1, 2, \dots, \ell\}$. The disjoint union of all paths P_i is a forest $F = P_1 \oplus P_2 \oplus \dots \oplus P_\ell$ contained in T . By Lemma 5, every forest admits an odd edge coloring by at most 2 colors, say 1 and 2. Observe also that the graph $G'_u - F$ is an odd graph, which we can color by color 3. Thus, we infer that G'_u admits an odd edge coloring by at most 3 colors.

Assume now that n is odd. The graph $G'_u - v$ is a connected graph with an even number of vertices, hence $T - v$ contains a forest F such that the graph $G'_u - v - F$ is an odd graph, which we color by color 3. If the degree of v is even in G'_u , we add the edge of T incident to v to the forest F . As in the previous case, we can color F by at most 2 colors, 1 and 2. It remains to consider the edges incident to v that are not colored yet. Recall that all the edges incident to v in G'_u are incident to u . The number of such edges is odd, so we color them by color 4, obtaining an odd edge coloring of G'_u with at most 4 colors.

Notice that in both graphs, G'_u and G'_v , we colored the k edges between u and v similarly. If k is odd, all the edges admit the same color, and in case when k is even, one edge is colored by one color and all the rest by some other color. Hence, there exists a permutation of the colors of G'_v such that the edges between u and v receive the same colors in both graphs. Notice that both colorings together induce an odd edge coloring of G by at most 4 colors which establishes the lemma. \square

We will make use of the following observation presented in [3].

Proposition 7. *Let G be a loopless multigraph with an odd number of edges between any pair of adjacent vertices. Then,*

$$\chi'_o(G) \leq 4.$$

Now, we prove Theorem 2 in a similar manner as in [3].

Proof of Theorem 2. Let $G = (V, E, F)$ be a 2-edge connected plane graph. We will find an FPE-coloring of G with 16 colors in two phases. First, let φ be a quasi-facially-odd edge coloring of G with 4 colors, which exists by Lemma 3. In the second phase, we granulate the coloring φ in such a way that every color appears odd or zero times on any facial walk in G .

Let C be the set of 4 colors used in φ . For every color $c \in C$ let G_c^* be the graph with the vertex set $V(G_c^*) = V(G^*)$ and the edge set $E(G_c^*)$ being a subset of the edges $E(G^*)$, where G^* is the dual of G . In particular, the edge e^* of G^* is in G_c^* if the color c was assigned to the edge e associated to e^* . Since G is bridgeless, its dual G^* is loopless and so is G_c^* , for any color c . Moreover, any two adjacent vertices of G_c^* are connected by an odd number of edges, except for the pairs of vertices representing the faces incident to C_5 -blocks in G , which are connected with two edges. Notice that the two edges form a 2-bridge in G_c^* , which separates the component in the interior of a C_5 -block from the component in the exterior.

By Proposition 7 and Lemma 6, there exists an odd edge coloring ω of the edges of G_c^* by at most 4 colors. Now, we replace the color c on the edge e with the color (c, i) , where $i = \omega(e^*)$. The odd or zero degree of every vertex in G_c^* assures that every color (c, i) appears odd number of times or not at all on the boundary of every face of G . Hence, after the procedure described above is performed on all 4 colors of C , we obtain an FPE-coloring of G with at most 16 colors. \square

3 Conclusion

Notice that our bound holds for graphs embedded on other surfaces if no face is incident to itself. Such a case results in a multigraph with loops when defining the dual G_c^* restricted to one color c . Whenever there is a component with one vertex incident to several loops in a graph, it cannot be partitioned into edge-disjoint subgraphs.

In the introduction we mention that there exist graphs that need at least 10 colors for an FPE-coloring, hence there remains a gap of 6 colors between the upper and lower bound of the facial parity chromatic index of 2-edge connected plane graphs. In the proof we use a combination of facially-odd edge coloring and partition the edges of every color class into odd subgraphs using 4 colors for each color. However, the fourth color in the odd edge coloring appears only on the edges incident to one vertex. Therefore, we believe that the argument could be modified in such a way that the fourth color could be omitted if the first phase of the coloring is carefully performed.

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